# Local definitions of formations of finite groups

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**Abstract**

A problem of constructing of local definitions for formations of finite groups is discussed in the article. The author analyzes relations between local definitions of various types. A new proof of existence of an $\omega$-composition satellite of an $\omega$-solubly saturated formation is obtained. It is proved that if a non-empty formation of finite groups is $\mathfrak{X}$-local by Förster, then it has an $\mathfrak{X}$-composition satellite.

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## 1. Introduction

We consider only finite groups. So, all group classes considered are subclasses of the class $\mathfrak{E}$ of all finite groups. Recall that a formation is a group class closed under taking homomorphic images and subdirect products (see [1]). A formation $\mathfrak{F}$ is said to be $p$-saturated ($p$ a prime) if the condition:

$$G/N \in \mathfrak{F} \text{ for a } G\text{-invariant } p\text{-subgroup } N \text{ of } \Phi(G)$$

always implies $G \in \mathfrak{F}$. A formation $\mathfrak{F}$ is said to be $\mathfrak{N}_p$-saturated if the condition

$$G/\Phi(N) \in \mathfrak{F} \text{ for a normal } p\text{-subgroup } N \text{ of } G$$

always implies $G \in \mathfrak{F}$.

If a formation is $p$-saturated for any prime $p$, then it is called saturated. Clearly, every $p$-saturated formation is $\mathfrak{N}_p$-saturated. The converse is not true:

there is an extensive class of $\mathfrak{N}_p$-saturated formations which are not $p$-saturated. However, as it is established in [2], between local definitions of these two types of formations there is a close connection.

The concept of local definitions of saturated formations was considered for the first time by W. Gaschütz [1]. Following [3], we formulate it in the general form.

A local definition is a map $f{:}\mathfrak{E} \to \{\text{formations}\}$ together with a $f$-rule which decide whether a chief factor is $f$-central or $f$-eccentric in a group. In addition, we follow the agreement that the local definition $f$ does not distinguish between non-identity groups with the same (up to isomorphism) set of composition factors. Therefore, for any fixed prime $p$, $f$ is not distinguish between any two non-identity $p$-groups; we will denote through $f(p)$ a value of $f$ on non-identity $p$-groups.

If a class $\mathfrak{F}$ coincides with the class of all groups all of whose chief factors are $f$-central, we say that $f$ is a local definition of $\mathfrak{F}$. It generalises the concept of nilpotency. Thus, the problem of finding local definitions for group classes is equivalent to a problem of finding classes of generalized nilpotent groups.

In this paper we analyze relations between local definitions of different types and give a new proof of a theorem on a local definition of a formation which is $\mathfrak{N}_p$-saturated for any $p$ in a set $\omega$ of primes.

## 2. Preliminaries

We use standard notations and definitions [4]. We say that a map $f$ does not distinguish between $\mathfrak{H}$-groups if $f(A) = f(B)$ for any two groups $A$ and $B$ in $\mathfrak{H}$. Following Gaschütz, the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of a group $G$ is the least normal subgroup with quotient in $\mathfrak{F}$. The Gaschütz product $\mathfrak{F} \circ \mathfrak{H}$ of formations $\mathfrak{F}$ and $\mathfrak{H}$ is defined as the class of all groups $G$ such that $G^{\mathfrak{H}} \in \mathfrak{F}$. If $\mathfrak{F}$ is closed under taking of normal subgroups, then $\mathfrak{F} \circ \mathfrak{H}$ coincides with the class $\mathfrak{F}\mathfrak{H}$ of all extensions of $\mathfrak{F}$-groups by $\mathfrak{H}$-groups.

$\mathbb{P}$ is the set of all primes; $\mathrm{Char}(\mathfrak{X})$ is the set of orders of all simple abelian groups in $\mathfrak{X}$. A group $G$ is called a $pd$-group if its order is divisible by a prime $p$; $C_p$ is a group of order $p$; if $\omega \subseteq \mathbb{P}$, then $\omega' = \mathbb{P} \setminus \omega$; an $\omega d$-group (a chief $\omega d$-factor) is a group (a chief factor) being $pd$-group for some $p \in \omega$; $G_{\omega d}$ is the largest normal subgroup all of whose $G$-chief factors are $\omega d$-groups ($G_{\omega d} = 1$ if all minimal normal subgroups in $G$ are $\omega'$-groups). If $\mathfrak{H}$ is a class of groups, then $\mathfrak{H}_\omega$ is the class of all $\omega$-groups in $\mathfrak{H}$. A chief factor $H/K$ of $G$ is called a chief $\mathfrak{H}$-factor if $H/K \in \mathfrak{H}$. The socle $\mathrm{Soc}(G)$ of a group $G \neq 1$ is the product

of all minimal normal subgroups of $G$.

$[A]B$ is a semidirect product with a normal subgroup $A$; $O_\omega(G)$ is the largest normal $\omega$-subgroup in $G$; $\pi(G)$ is the set of all primes dividing the order of a group $G$; $\pi(\mathfrak{F}) = \cup_{G\in\mathfrak{F}}\pi(G)$; $\mathfrak{N}$ is the class of all nilpotent groups; $\mathfrak{A}$ is the class of all abelian groups; $\mathrm{Com}(G)$ is the class of all groups that are isomorphic to composition factors of a group $G$; $\mathrm{Com}(\mathfrak{F}) = \cup_{G\in\mathfrak{F}}\mathrm{Com}(G)$; $\mathrm{Com}^+(\mathfrak{F})$ is the class of all abelian groups in $\mathrm{Com}(\mathfrak{F})$; $\mathrm{Com}^-(\mathfrak{F})$ is the class of all non-abelian groups in $\mathrm{Com}(\mathfrak{F})$; $(G)$ is the class of all groups isomorphic to $G$; $\mathfrak{J}$ is the class of all simple (abelian and non-abelian) groups; if $\mathfrak{L}$ is a subclass in $\mathfrak{J}$, then $\mathfrak{L}' = \mathfrak{J}\setminus\mathfrak{L}$; $\mathfrak{L}^+$ is the class of all abelian groups in $\mathfrak{L}$, $\mathfrak{L}^- = \mathfrak{L}\setminus\mathfrak{L}^+$. $\mathrm{E}\mathfrak{H}$ is the class of all groups $G$ such that $\mathrm{Com}(G)\subseteq\mathfrak{H}$; $G_{\mathrm{E}\mathfrak{H}}$ is the $\mathrm{E}\mathfrak{H}$-radical of $G$, the largest normal $\mathrm{E}\mathfrak{H}$-subgroup in $G$. If $S\in\mathfrak{J}$, then $C^S(G)$ is the intersection of centralizers of all chief $\mathrm{E}(S)$-factors of $G$ ($C^S(G) = G$ if $S\notin\mathrm{Com}(G)$); if $S = C_p$, we write $C^p(G)$ in place of $C^S(G)$.

**Lemma 2.1** (see [3], Lemmas 2–3). $(a)$ *If $S$ is a non-abelian simple group, then $C^S(G)$ is the $\mathrm{E}(S)'$-radical of $G$, the largest normal subgroup not having composition factors isomorphic to $S$.*

$(b)$ *Let $p$ be a prime, and $\mathfrak{H}$ be the class of all groups all of whose chief $p$-factors are central. Then $C^p(G)$ is the $G_{\mathfrak{H}}$-radical of $G$, for every group $G$.*

The following three lemmas are reformulations of Lemmas IV.4.14–IV.4.16 in [4] whose proofs use only $p$-solubly saturation.

**Lemma 2.2.** *Let $\mathfrak{F}$ be an $\mathfrak{N}_p$-saturated formation, $p$ a prime. If $C_p\in\mathrm{Com}(\mathfrak{F})$, then $\mathfrak{N}_p\subseteq\mathfrak{F}$.*

**Lemma 2.3.** *Let $\mathfrak{F}$ be an $\mathfrak{N}_p$-saturated formation containing $\mathfrak{N}_p$, $p$ a prime. Let $N$ be an elementary abelian normal $p$-subgroup in $G$ such that $[N](G/N)\in\mathfrak{F}$. Then $G\in\mathfrak{F}$.*

**Lemma 2.4.** *Let $p$ be a prime, and let $\mathfrak{F}$ be an $\mathfrak{N}_p$-saturated formation containing $\mathfrak{N}_p$. Let $N$ be an elementary abelian normal $p$-subgroup in $G$ such that $G/N\in\mathfrak{F}$ and $[N](G/C_G(N))\in\mathfrak{F}$. Then $G\in\mathfrak{F}$.*

**Proof.** Set $M = [N](G/N)$, $C = C_G(N)$. Evidently, $C/N = C_{G/N}(N)$. In the group $M$ we have $C_M(N) = N\times C/N$ and $C/N$ is normal in $M$. Hence $M/(C/N)\simeq[N](G/C)\in\mathfrak{F}$. Since $M/N\in\mathfrak{F}$, it follows that $M/N\cap(C/N)\simeq M\in\mathfrak{F}$. Now we apply Lemma 2.3. □

**Lemma 2.5.** *Let $\mathfrak{F}$ be an $\mathfrak{N}_p$-saturated formation containing $\mathfrak{N}_p$, $p$ a prime.*

*Let $H \in \mathfrak{F}$ and let $C^p(H) \leq L \trianglelefteq H$. If $N$ is an irreducible $\mathbb{F}_p(H/L)$-module, then $[N](H/L) \in \mathfrak{F}$.*

**Lemma 2.6** (see [4], Proposition IV.1.5). *Let $\mathfrak{F}$ be a formation and $G \in \mathfrak{F}$. Let $S, R, K$ be normal subgroups in $G$ such that $S \subseteq R$ and $K \subseteq C_G(R/S)$. Then $[R/S](G/K) \in \mathfrak{F}$.*

**Lemma 2.7** (see [5] or [6], Theorem 7.11). *If $H/\Phi(G) = \mathrm{Soc}(G/\Phi(G)$, then $C_G(H) \subseteq H$.*

**Lemma 2.8** (see [4], Lemma IV.4.11). *Let $p$ be a prime, $L = \Phi(O_p(G))$. Then $C^p(G/L) = C^p(G)/L$.*

## 3. Local and $\omega$-local satellites

The following type of a local definition was proposed by W. Gaschütz [1].

**Definition 3.1.** Let $f$ be a local definition such that

$$f(A) = \bigcap_{p \in \pi(A)} f(p)$$

for any group $A \neq 1$. Let an $f$-rule be defined as follows: a chief factor $H/K$ of a group $G$ is $f$-central if $G/C_G(H/K) \in f(H/K)$. Then $f$ is called *a local satellite*.

**Definition 3.2** (see [4], p. 387). Let $A$ be a group of operators for a group $G$, and $f$ a local satellite.

(i) We say that $A$ acts $f$-*centrally* on an $A$-composition factor $H/K$ of $G$ if $A/C_A(H/K) \in f(p)$ for every prime $p \in \pi(H/K)$.

(ii) We say that $A$ acts $f$-*hypercentrally* on $G$ if $A$ acts $f$-centrally on every $A$-composition factor of $G$.

The convenient notation $LF(f)$ for a group class with a local satellite $f$ was introduced by Doerk and Hawkes [4]. Clearly, $LF(f)$ is a non-empty formation (we have always $1 \in LF(f)$).

The following proposition is evident.

**Proposition 3.1.** *Let $f$ be a local satellite and $\pi = \{p \in \mathbb{P} \mid f(p) \neq \varnothing\}$. Then $LF(f)$ consists precisely of $\pi$-groups $G$ satisfying the following condition: $G/O_{p',p}(G) \in f(p)$ for any $p \in \pi(G)$. Thus, if $\pi = \varnothing$, we have $LF(f) = (1)$. If $\pi \neq \varnothing$, we have that*

$$LF(f) = \mathfrak{E}_\pi \bigcap \Big( \bigcap_{p \in \pi} (\mathfrak{E}_{p'} \mathfrak{E}_p f(p)) \Big).$$

We remember the reader that a formation $\mathfrak{F}$ is saturated if $G/\Phi(G) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$ (by definition, the empty set is a saturated formation). W. Gashütz has shown that every formation with a local satellite is saturated. This fact follows also from the following theorem of P. Schmid.

**Theorem 3.1** (see [4], Theorem IV.6.7). *Let $f$ be a local satellite, and let $A$ be a group of operators for a group $G$. If $A$ acts $f$-hypercentrally on $G/\Phi(G)$, then $A$ acts likewise on $G$.*

The following remarkable result is known as the Gaschütz–Lubeseder–Schmid theorem, see [4], Theorem IV.4.6.

**Theorem 3.2.** *A non-empty formation has a local satellite if and only if it is saturated.*

It is straightforward to verify that if $\mathfrak{F}$ is a non-empty formation, then $\mathfrak{N}\mathfrak{F}$ is a formation with a local satellite $f$ such that $f(p) = \mathfrak{F}$ for every prime $p$. Evidently, the formation $\mathfrak{A}_p \times \mathfrak{N}_{p'}$ of all nilpotent groups with an abelian Sylow $p$-subgroup is not saturated, but for every prime $q \neq p$, $G/(\Phi(G) \cap O_q(G)) \in \mathfrak{A}_p \times \mathfrak{N}_{p'}$ always implies $G \in \mathfrak{A}_p \times \mathfrak{N}_{p'}$. One more fact of the same sort is the following. Consider a saturated formation of the form $\mathfrak{M} \circ \mathfrak{H}$. Here $\mathfrak{H}$ can be non-saturated, but for every prime $p \in \mathbb{P} \setminus \pi(\mathfrak{M})$, $G/(\Phi(G) \cap O_p(G)) \in \mathfrak{H}$ always implies $G \in \mathfrak{H}$. The facts of such kind lead to the concept of a $\omega$-saturated formation [11].

**Definition 3.3.** Let $\omega$ be a set of primes. A formation $\mathfrak{F}$ is called *$\omega$-saturated* if for every prime $p \in \omega$, $G/(\Phi(G) \cap O_p(G)) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$.

The problem of finding of local definitions of $\omega$-saturated formations was considered in [7] and [3]. While solving this problem the following concept of small centralizer was useful (see [8]).

**Definition 3.4.** Let $H/K$ be a chief factor of a group $G$. The *small centralizer* $c_G(H/K)$ of $H/K$ in $G$ is the subgroup generated by all normal subgroups $N$ of $G$ such that $\mathrm{Com}(NK/K) \cap \mathrm{Com}(H/K) = \varnothing$.

With the help of Definition 3.4 we can introduce the concept '$\omega$-saturated satellite' as follows.

**Definition 3.5.** Let $\omega$ be a set of primes, and $f$ a local definition which does not distinguish between all non-identity $\omega'$-groups; if $\omega' \neq \varnothing$, we denote

through $f(\omega')$ a value of $f$ on non-identity $\omega'$-groups. In addition, we assume that

$$f(A) = \bigcap_{p \in \pi(A) \cap \omega} f(p)$$

for any $\omega d$-group $A$. Let an $f$-rule be defined by the following way: a chief factor $H/K$ of $G$ is $f$-central in $G$ if either $H/K$ is an $\omega d$-group and $G/C_G(H/K) \in f(H/K)$ or else $H/K$ is an $\omega'$-group and $G/c_G(H/K) \in f(\omega')$. Then $f$ is called an $\omega$-*local satellite*. We denote by $LF_\omega(f)$ the class of all groups all of whose chief factors are $f$-central. By definition, $1 \in LF_\omega(f)$.

Clearly, if $\omega = \mathbb{P}$, then an $\omega$-local satellite $f$ is a local satellite, and $LF_\omega(f) = LF(f)$. If $\omega \neq \mathbb{P}$ and $f(\omega') = \varnothing$, then $LF_\omega(f) = LF(h)$ where $h(p) = f(p)$ if $p \in \omega$, and $h(p) = \varnothing$ if $p \in \omega'$.

**Lemma 3.1** (see [3], Lemma 1). *Let $\mathfrak{L}$ be a subclass in $\mathfrak{J}$, and $\{S_i \mid i \in I\}$ be the set of all $\mathrm{E}\mathfrak{L}$-factors of a group $G$. Then $\bigcap_{i \in I} c_G(S_i)$ is the $\mathrm{E}(\mathfrak{L}')$-radical $G_{\mathrm{E}(\mathfrak{L}')}$ of $G$.*

**Remark 3.1.** In Lemma 3.1 the set $\{c_G(S_i) \mid i \in I\}$ can be empty. We always follow the agreement that the intersection of an empty set of subgroups of $G$ coincides with $G$.

The following proposition is similar to Proposition 3.1.

**Proposition 3.2.** *Let $f$ be an $\omega$-local satellite, and $\omega$ a proper subset in $\mathbb{P}$. Let $\pi = \{p \in \omega \mid f(p) \neq \varnothing\}$. Then:*

(1) *if $\pi = \varnothing$ and $f(\omega') = \varnothing$, then $LF_\omega(f) = (1)$;*

(2) *if $\pi = \varnothing$ and $f(\omega') \neq \varnothing$, then $LF_\omega(f) = \mathfrak{E}_{\omega'} \cap f(\omega')$;*

(3) *if $f(\omega') \neq \varnothing$, then $LF_\omega(f)$ consists precisely of groups $G$ such that $G/G_{\omega d} \in f(\omega')$ and $G/O_{p',p}(G) \in f(p)$ for any $p \in \pi(G) \cap \omega$.*

**Proof.** Statements (1) and (2) are evident.

Prove (3). Assume that $f(\omega') \neq \varnothing$, and let $G \in LF_\omega(f)$. Let $\mathfrak{T}$ be the set of all chief $\omega'$-factors in $G$. If a chief factor $H/K$ of $G$ is an $\omega'$-group, then $G/c_G(H/K) \in f(\omega')$. Therefore, $G/\bigcap_{H/K \in \mathfrak{T}} c_G(H/K) \in f(\omega')$. By Lemma 3.1, $\bigcap_{H/K \in \mathfrak{T}} c_G(H/K) = G_{\omega d}$. So, $G/G_{\omega d} \in f(\omega')$. If $p \in \omega$ and $H/K$ is an chief $pd$-factor, then $G/C_G(H/K) \in f(p)$, and we have $G/O_{p',p}(G) \in f(p)$.

Conversely, let $G$ be a group such that $G/G_{\omega d} \in f(\omega')$ and $G/O_{p',p}(G) \in f(p)$ for any $p \in \pi(G) \cap \omega$. Clearly, we have that all $G$-chief $\omega d$-factors are $f$-central. Let $H/K$ be a $G$-chief $\omega'$-factor of $G$. Then $G_{\omega d}K/K \subseteq c_G(H/K)$, and $G/G_{\omega d} \in f(\omega')$ implies $G/c_G(H/K) \in f(\omega')$. □

The following result extends Theorem 3.2 to $\omega$-saturated formations.

**Theorem 3.3** (see [7], Theorem 1). *Let $\omega$ be a set of primes. A non-empty formation has a $\omega$-local satellite if and only if it is $\omega$-saturated.*

## 4. Composition and $\mathfrak{L}$-composition satellites

Gaschütz's main idea [1] was to study groups modulo $p$-groups, and he implemented it through local satellites of soluble formations. While considering non-soluble formations, we have to follow the following principle: study groups modulo $p$-groups and simple groups. That approach was proposed in the lecture [9] at the conference in 1973; in that lecture composition satellites were considered under the name 'primarily homogeneous screens'.

**Definition 4.1.** Let $f$ be a local definition, and let an $f$-rule be defined as follows: a chief factor $H/K$ of a group $G$ is $f$-central if $G/C_G(H/K) \in f(H/K)$. Then $f$ is called *a composition satellite*. We denote by $CF(f)$ the class of all groups all of whose chief factors are $f$-central.

**Definition 4.2.** Let $A$ be a group of operators for a group $G$, and $f$ a composition satellite.

(i) We say that $A$ acts *$f$-centrally* on an $A$-composition factor $H/K$ of $G$ if $A/C_A(H/K) \in f(H/K)$.

(ii) We say that $A$ acts *$f$-hypercentrally* on $G$ if it acts $f$-centrally on every $A$-composition factor of $G$.

As an example, we consider the class $\mathfrak{N}^*$ of all quasinilpotent groups (for the definition of a quasinilpotent group, see [12], Definition X.13.2). It is easy to check that $\mathfrak{N}^* = CF(f)$ where $f$ is a composition satellite such that $f(p) = (1)$ for every prime $p$, and $f(S) = \text{form}(S)$ for every non-abelian simple group $S$. Here $\text{form}(S)$ is a least formation containing $S$; it consists of all groups represented as a direct product $A_1 \times \cdots \times A_n$ with $A_i \simeq S$ for any $i$. The formation $\mathfrak{N}^*$ is non-saturated, but it is solubly saturated.

As pointed out in [4], formations with composition satellites were also considered—in different terminology—by R. Baer in his unpublished manuscript. By R. Baer, a formation $\mathfrak{F}$ is called *solubly saturated* if the condition $G/\Phi(G_{\mathfrak{S}}) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$ (here $G_{\mathfrak{S}}$ is the soluble radical of $G$). The question of the coincidense of the family of non-empty solubly saturated formations and the family of formations with composition satellites was solved by the following result due to R. Baer.

**Theorem 4.1** (see [4], Theorem IV.4.17). *A non-empty formation has a composition satellite if and only if it is solubly saturated.*

A composition satellite $h$ is called *integrated* if $h(S) \subseteq CF(h)$ for any simple group $S$. If $\mathfrak{F} = CF(f)$, then $\mathfrak{F} = CF(h)$ where $h(S) = f(S) \cap \mathfrak{F}$ for any simple group $S$. Thus, if a formation has a composition satellite, then it has an integrated composition satellite.

**Remark 4.1.** Let $\{CF(f_i) \mid i \in I\}$ be a family of formations having composition satellites. Let $f = \cap_{i\in I} f_i$ be a composition satellite such that $f(S) = \cap_{i\in I} f_i(S)$ for every $S \in \mathfrak{J}$. Clearly, $CF(f) = \cap_{i\in I} CF(f_i)$.

**Remark 4.2.** Let $\mathfrak{X}$ be a set of groups. Let $\{\mathfrak{F}_i \mid i \in I\}$ be the class of all formations $\mathfrak{F}_i$ satisfying the following two conditions: 1) $\mathfrak{X} \subseteq \mathfrak{F}_i$; 2) $\mathfrak{F}_i$ has a composition satellite. Set $\mathrm{cform}(\mathfrak{X}) = \cap_{i\in I} \mathfrak{F}_i$. By Remark 4.1, $\mathrm{cform}(\mathfrak{X})$ has a composition satellite. In the subsequent we will use that notation $\mathrm{cform}(\mathfrak{X})$.

**Remark 4.3.** Assume that a non-empty formation $\mathfrak{F}$ has an composition satellite. Let $\{f_i \mid i \in I\}$ be the class of all composition satellites of $\mathfrak{F}$. Having in mind Remarks 4.1 and 4.2 we see that $f = \cap_{i\in I} f_i$ is a composition satellite of $\mathfrak{F}$; $f$ is called the *minimal composition satellite* of $\mathfrak{F}$.

**Lemma 4.1.** *Let $\mathfrak{X}$ be a set of groups, and $S$ a simple group. Then $\mathfrak{H} = \mathrm{Q}(G/C^S(G) \mid G \in \mathrm{form}(\mathfrak{X}))$ is a formation, and $\mathrm{Com}(\mathfrak{H}) \subseteq \mathrm{Com}(\mathfrak{X})$.*

**Proof.** By Proposition IV.1.10 in [4], $\mathfrak{H}$ is a formation. By Lemma II.1.18 in [4], $\mathrm{form}(\mathfrak{X}) = \mathrm{QR}_0\mathfrak{X}$. Therefore, inclusion $\mathrm{Com}(\mathfrak{H}) \subseteq \mathrm{Com}(\mathfrak{X})$ is valid. □

**Lemma 4.2.** *Let $\mathfrak{X}$ be a non-empty set of groups, and $f$ be a composition satellite such that $f(S) = \mathrm{Q}(G/C^S(G) \mid G \in \mathrm{form}(\mathfrak{X}))$ if $S \in \mathrm{Com}(\mathfrak{X})$, and $f(S) = \varnothing$ if $S \in \mathfrak{J} \setminus \mathrm{Com}(\mathfrak{X})$. Then $f$ is the minimal composition satellite of* $\mathrm{cform}(\mathfrak{X})$.

**Proof.** Let $f_1$ be the minimal composition satellite of $\mathfrak{F} = \mathrm{cform}(\mathfrak{X})$ (see Remark 4.3). We will prove that $f_1 = f$.

Since $\mathfrak{X} \subseteq \mathfrak{F}$, $G/C^S(G) \in f_1(S)$ for any group $G \in \mathfrak{X}$ and any $S \in \mathrm{Com}(G)$ and therefore $f(S) \subseteq f_1(S)$. So $CF(f) \subseteq \mathfrak{F} \subseteq CF(f_1)$. On the other hand, $\mathfrak{X} \subseteq CF(f)$. Thus $\mathfrak{F} = CF(f)$ and $f = f_1$. □

The following theorem proved independently in [13] and [14] was the first important result on composition formations.

**Theorem 4.2.** *Let $f$ be an integrated composition satellite. Let $A$ be a group of automorphisms of a group $G$. If $A$ acts $f$-hypercentrally on $G$, then $A \in CF(f)$.*

Applying Theorem 4.2 to the formation $\mathfrak{U}$ of all supersoluble groups, we have the following result.

**Theorem 4.3** (see [13], Theorem 2.4). *Let $A$ be a group of automorphisms of a group $G$. Assume that there exists a chain of $A$-invariant subgroups*

$$G = G_0 > G_1 > \cdots > G_n = 1$$

*with prime indices $|G_{i-1} : G_i|$. Then $A$ is supersoluble.*

In 1968 S.A. Syskin tried to prove Theorem 4.3 in the soluble universe, but his proof [15] is false.

In [2] there has been begun studying of local definitions of $\omega$-solubly saturated formations.

**Definition 4.3.** Let $\omega$ be a set of primes. A formation $\mathfrak{F}$ is called:

(1) $\omega$-*solubly saturated* if the condition

$$G/N \in \mathfrak{F} \text{ for } G\text{-invariant } \omega\text{-subgroup } N \text{ in } \Phi(G_{\omega\text{-}\mathfrak{S}})$$

always implies $G \in \mathfrak{F}$ (here $G_{\omega\text{-}\mathfrak{S}}$ is the $\omega$-soluble radical of $G$);

(2) $\mathfrak{N}_\omega$-*saturated* if for every prime $p \in \omega$, the condition $G/\Phi(O_p(G)) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$.

Later we will establish that the $p$-solubly saturation is equivalent to the $\mathfrak{N}_p$-saturation, and therefore a formation $\mathfrak{F}$ is $\omega$-solubly saturated if and only if it is $p$-solubly saturated for every $p \in \omega$.

**Definition 4.4.** Let $\mathfrak{L}$ be a class of simple groups. Let $f$ be a local definition which does not distinguish between all non-identity $\mathrm{E}(\mathfrak{L}')$-groups; if $\mathfrak{L}' \neq \varnothing$, we denote by $f(\mathfrak{L}')$ an value of $f$ on non-identity $\mathrm{E}(\mathfrak{L}')$-groups. Let $f$-rule be defined as follows: a chief factor $H/K$ of a group $G$ is $f$-central in $G$ if either $H/K$ is an $\mathrm{E}\mathfrak{L}$-group and $G/C_G(H/K) \in f(H/K)$ or $H/K$ is a $\mathrm{E}(\mathfrak{L}')$-group and $G/c_G(H/K) \in f(H/K) = f(\mathfrak{L}')$. Then $f$ is called an $\mathfrak{L}$-*composition satellite.* We denote by $CF_{\mathfrak{L}}(f)$ the class of all groups all of whose chief factors are $f$-central. By definition, $1 \in CF_{\mathfrak{L}}(f)$.

Clearly, if $\mathfrak{L} = \mathfrak{J}$, then an $\mathfrak{L}$-composition satellite $f$ is a composition satellite, and $CF_{\mathfrak{L}}(f) = CF(f)$. If $\mathfrak{L} \neq \mathfrak{J}$ and $f(\mathfrak{L}') = \varnothing$, then $CF_{\mathfrak{L}}(f) = CF(h)$ where $h(S) = f(S)$ if $S \in \mathfrak{L}$, and $h(S) = \varnothing$ if $S \in \mathfrak{L}'$.

**Proposition 4.1.** *Let $\mathfrak{L}$ be a class of simple groups, and $f$ an $\mathfrak{L}$-composition satellite. Let $\mathfrak{K} = \{S \in \mathfrak{L} \mid f(S) \neq \varnothing\}$. Then:*

(1) *if $\mathfrak{K} = \varnothing$ and $f(\mathfrak{L}') = \varnothing$, then $CF_{\mathfrak{L}}(f) = (1)$;*

(2) *if $\mathfrak{K} = \varnothing$ and $f(\mathfrak{L}') \neq \varnothing$, then $CF_{\mathfrak{L}}(f) = \mathrm{E}(\mathfrak{L}') \cap f(\mathfrak{L}')$;*

(3) *if $f(\mathfrak{L}') \neq \varnothing$, then $CF_{\mathfrak{L}}(f)$ consists precisely of groups $G$ such that $G/G_{\mathrm{E}\mathfrak{L}} \in f(\mathfrak{L}')$ and $G/C^S(G) \in f(S)$ for every $S \in \mathrm{Com}(G) \cap \mathfrak{L}$.*

**Proof.** Statements (1) and (2) are evident.

Prove (3). Assume that $f(\mathfrak{L}') \neq \varnothing$, and let $G \in CF_{\mathfrak{L}}(f)$. Let $\mathfrak{T}$ be the set of all chief $\mathrm{E}(\mathfrak{L}')$-factors in $G$. If a chief factor $H/K$ of $G$ is an $\mathrm{E}(\mathfrak{L}')$-group, then $G/c_G(H/K) \in f(\mathfrak{L}')$. Therefore, $G/\bigcap_{H/K\in\mathfrak{T}} c_G(H/K) \in f(\mathfrak{L}')$. By Lemma 3.1, $\bigcap_{H/K\in\mathfrak{T}} c_G(H/K) = G_{\mathrm{E}\mathfrak{L}}$. So, $G/G_{\mathrm{E}(\mathfrak{L})} \in f(\mathfrak{L}')$. If $S \in \mathfrak{L}$ and $H/K$ is an chief $\mathrm{E}(\mathfrak{L})$-factor, then $G/C_G(H/K) \in f(S)$, and we have $G/C^S(G) \in f(S)$.

Conversely, let $G$ be a group such that $G/G_{\mathrm{E}\mathfrak{L}} \in f(\mathfrak{L}')$ and $G/C^S(G) \in f(S)$ for every $S \in \mathrm{Com}(G) \cap \mathfrak{L}$. Clearly, all chief $\mathrm{E}\mathfrak{L}$-factors of $G$ are $f$-central. Let $H/K$ be a chief $\mathrm{E}(\mathfrak{L}')$-factor of $G$. Then $G_{\mathrm{E}\mathfrak{L}} \subseteq c_G(H/K)$, and therefore $G/G_{\mathrm{E}\mathfrak{L}} \in f(\mathfrak{L}')$ implies $G/c_G(H/K) \in f(\mathfrak{L}')$. $\square$

An $\mathfrak{L}$-composition satellite $f$ is called *integrated* if $f(S) \in CF_{\mathfrak{L}}(f)$ for every $S \in \mathfrak{J}$. If $\mathfrak{F} = CF_{\mathfrak{L}}(f)$, then $\mathfrak{F} = CF_{\mathfrak{L}}(h)$ where $h(S) = f(S) \cap \mathfrak{F}$ for any simple group $S$. Thus, if a formation has an $\mathfrak{L}$-composition satellite, then it has an integrated $\mathfrak{L}$-composition satellite.

**Lemma 4.3.** *If $\mathfrak{F} = CF_{\mathfrak{L}}(f)$, then $\mathfrak{F} = CF_{\mathfrak{L}}(h)$ where $h$ is an integrated $\mathfrak{L}$-composition satellite such that $h(S) = \mathfrak{F}$ for every $S \in (\mathfrak{L}^+)'$.*

**Proof.** We can assume without loss of generality that $f$ is integrated. Let $\mathfrak{H} = CF_{\mathfrak{L}}(h)$ where $h(S) = f(S)$ if $S \in \mathfrak{L}^+$, and $h(S) = \mathfrak{F}$ if $S \in (\mathfrak{L}^+)'$. Evidently, $\mathfrak{F} \subseteq \mathfrak{H}$. Assume that $\mathfrak{H} \not\subseteq \mathfrak{F}$, and choose a group $G$ of minimal order in $\mathfrak{H} \setminus \mathfrak{F}$. Then $L = G^{\mathfrak{F}}$ is a unique minimal normal subgroup in $G$, and $L$ is not $f$-central. Clearly, $c_G(L) = 1$, and $C_G(L) = 1$ if $L$ is non-abelian. Let $A \in \mathrm{Com}(L)$. Applying Definition 4.4 and considering the cases $A \in \mathfrak{L}^+$, $A \in \mathfrak{L}^-$ and $A \in \mathfrak{L}'$, we arrive at a contradiction. $\square$

**Theorem 4.4** (see [16], Theorem 2). *Let $\mathfrak{F}$ be a non-empty formation, $\mathfrak{L}$ a class of simple groups. The following statements are equivalent:*

(1) *$\mathfrak{F}$ has an $\mathfrak{L}$-composition satellite;*

(2) *$\mathfrak{F}$ has an $\mathfrak{L}^+$-composition satellite.*

**Proof**. (1) $\Rightarrow$ (2). Let $\mathfrak{F} = CF_{\mathfrak{L}}(f)$. Applying Lemma 4.3 we can suppose

that $f$ is integrated and $f(S) = \mathfrak{F}$ for every $S \in (\mathfrak{L}^+)'$. Let $\mathfrak{H} = CF_{\mathfrak{L}^+}(h)$ where $h$ is an $\mathfrak{L}^+$-composition satellite such that $h(S) = f(S)$ if $S \in \mathfrak{L}^+$, and $h(S) = \mathfrak{F}$ if $S \in \mathfrak{L}' \cup \mathfrak{L}^- = (\mathfrak{L}^+)'$. We will prove that $\mathfrak{F} = \mathfrak{H}$.

If $G$ is a group of minimal order in $\mathfrak{F} \setminus \mathfrak{H}$, then $L = G^{\mathfrak{H}}$ is a unique minimal normal subgroup in $G$, and $L$ is not $h$-central. Clearly, $c_G(L) = 1$, and $C_G(L) = 1$ if $L$ is non-abelian. Applying Definition 4.4 we see that $L$ is $h$-central, a contradiction. Thus $\mathfrak{F} \subseteq \mathfrak{H}$.

Let $G$ be a group of minimal order in $\mathfrak{H} \setminus \mathfrak{F}$. Then $L = G^{\mathfrak{F}}$ is a unique minimal normal subgroup in $G$, and $L$ is not $f$-central. Clearly, $c_G(L) = 1$, and $C_G(L) = 1$ if $L$ is non-abelian. Applying again Definition 4.4 we see that $L$ is $f$-central, and we arrive at a contradiction. Thus $\mathfrak{H} \subseteq \mathfrak{F}$.

(2) $\Rightarrow$ (1). Let $\mathfrak{F} = CF_{\mathfrak{L}^+}(f)$. Applying Lemma 4.3 we can suppose that $f$ is integrated and $f((\mathfrak{L}^+)') = \mathfrak{F}$. Let $h$ be an $\mathfrak{L}$-composition satellite such that $h(S) = f(S)$ if $S \in \mathfrak{L}^+$, and $h(S) = \mathfrak{F}$ if $S \in (\mathfrak{L}^+)'$. It is easy to see that $\mathfrak{F} = CF_{\mathfrak{L}}(h)$. □

**Remark 4.4.** It follows from Theorem 4.4 that every non-empty formation $\mathfrak{F}$ with the property $\mathrm{Com}^+(\mathfrak{F}) \cap \mathfrak{L} = \varnothing$ has an $\mathfrak{L}$-composition satellite.

**Remark 4.5.** When $\mathfrak{L} = \mathfrak{L}^+$ and $\omega = \pi(\mathfrak{L})$, we usually use the term '$\omega$-composition satellite' and the notations $CF_\omega(f)$, $f(\omega')$ in place of the term '$\mathfrak{L}$-composition satellite' and the notations $CF_{\mathfrak{L}}(f)$, $f(\mathfrak{L}')$, respectively.

**Theorem 4.5** (see [2], Theorems 3.1 and 3.2). *Let $\mathfrak{F}$ be a non-empty formation, $\omega$ a set of primes. The following statements are pairwise equivalent:*

(1) *$\mathfrak{F}$ is $\mathfrak{N}_\omega$-saturated;*

(2) *$\mathfrak{F}$ is $\omega$-solubly saturated;*

(3) $\mathrm{cform}(\mathfrak{F}) \subseteq \mathfrak{N}_{\omega'}\mathfrak{F}$*;*

(4) *$\mathfrak{F} = CF_\omega(f)$ where $f$ is a $\omega$-composition satellite satisfying the following conditions:*

*(i) $f(p) = \mathrm{Q}(G/C^p(G) \mid G \in \mathfrak{F})$ if $p \in \omega$ and $C_p \in \mathrm{Com}(\mathfrak{F})$;*

*(ii) $f(p) = \varnothing$ if $p \in \omega$ and $C_p \notin \mathrm{Com}(\mathfrak{F})$;*

*(iii) $f(S) = \mathfrak{F}$ if $S \in \mathfrak{J} \setminus \{C_p \mid p \in \omega\}$.*

**Proof.** (1) $\Rightarrow$ (3). Set $\mathfrak{H} = \mathrm{cform}(\mathfrak{F})$. Fix $p \in \omega$. Since $\mathfrak{H} \subseteq \mathfrak{N}\mathfrak{F}$, it is sufficient to show that $\mathfrak{H} \subseteq \mathfrak{N}_{p'}\mathfrak{F}$. Let $G$ be a group of minimal order in $\mathfrak{H} \setminus \mathfrak{N}_{p'}\mathfrak{F}$. Clearly, $G$ is monolithic and $L = \mathrm{Soc}(G)$ is the $\mathfrak{N}_{p'}\mathfrak{F}$-residual of $G$. Since $\mathfrak{F} \subseteq \mathfrak{N}_{p'}\mathfrak{F}$, it follows that $G^{\mathfrak{F}} \geq L$. Since $G \in \mathfrak{H} \subseteq \mathfrak{N}\mathfrak{F}$, we have

$G^{\mathfrak{F}} \in \mathfrak{N}$. Since $G$ is monolithic and $G \notin \mathfrak{N}_{p'}\mathfrak{F}$, it follows that $G^{\mathfrak{F}}$ is a $p$-group. From $G/L \in \mathfrak{N}_{p'}\mathfrak{F}$ it follows that $(G/L)^{\mathfrak{F}} = G^{\mathfrak{F}}/L$ is a $p'$-group. Therefore, $G^{\mathfrak{F}} = L = G^{\mathfrak{N}_{p'}\mathfrak{F}}$. By Lemma 4.2, $\mathfrak{H}$ has a composition satellite $h$ such that $h(p) = \mathrm{Q}(A/C^p(A) \mid A \in \mathfrak{F})$. Since $L$ is a $p$-group, we have $C_p \in \mathrm{Com}(G)$. Now from Lemma 4.1 it follows that $C_p \in \mathrm{Com}(\mathfrak{F})$. Thus, applying Lemma 2.2, it follows that $\mathfrak{N}_p \subseteq \mathfrak{F}$. Since $h$ is a composition satellite of $\mathfrak{H}$, we have that $G/C_G(L) \in h(p)$. Therefore $[L](G/C_G(L)) \in \mathfrak{H}$, and $G/C_G(L)$ acts fixed-point-free on $L$. It follows that $G/C_G(L) \simeq T/N$, $T = A/C^p(A)$, $A \in \mathfrak{F}$. If $C_p \notin \mathrm{Com}(A)$, then $A = C_p(A)$, $T = 1$ and $G = L \in \mathfrak{F}$. Assume that $C_p \in \mathrm{Com}(A)$. Since $G/C_G(L) \simeq T/N$, we can consider $L$ as an irreducible $\mathbb{F}_p(T/N)$-module. Then $L$ becomes an irreducible $\mathbb{F}_pT$-module by inflation (see [4], p. 105). Since $T = A/C^p(A)$, we have by Lemma 2.5 that $[L]T \in \mathfrak{F}$. By Lemma 2.6 it then follows that $[L](T/N) \in \mathfrak{F}$. From this and $T/N \simeq G/C_G(L)$ we deduce that $[L](G/C_G(L) \in \mathfrak{F}$. Hence, by Lemma 2.4 it follows that $G \in \mathfrak{F}$.

(3) $\Rightarrow$ (2). It is sufficient to consider only the case $\omega = \{p\}$. Let $H$ be a $p$-soluble normal subgroup in $G$, $L = O_p(H) \cap \Phi(H)$, and $G/L \in \mathfrak{F}$. We need to prove that $G \in \mathfrak{F}$. If $O_{p'}(H) \neq 1$, then $LO_{p'}(H)/O_{p'}(H) \leq \Phi(H/O_{p'}(H))$, and by induction we have $G/O_{p'}(H) \in \mathfrak{F}$. From this and $G/L \in \mathfrak{F}$ it follows that $G \in \mathfrak{F}$. Assume that $O_{p'}(H) = 1$. By Lemma 4.2, $\mathfrak{H} = \mathrm{cform}(\mathfrak{F})$ has a composition satellite $h$ such that $h(p) = \mathrm{Q}(A/C^p(A) \mid A \in \mathfrak{F})$. Let $t$ be a local satellite such that $t(p) = h(p)$ and $t(q) = \mathfrak{E}$ for every prime $q \neq p$. Since $G/L \in \mathfrak{F} \subseteq \mathfrak{H}$ and $L = \Phi(H)$, $G$ acts $t$-hypercentrally on $H/\Phi(H)$. By Theorem 3.1, $G$ acts $t$-hypercentrally on $L = \Phi(H)$. But then $G$ acts $h$-hypercentrally on $L = \Phi(H)$, and we get $G \in \mathfrak{H} \subseteq \mathfrak{N}_{p'}\mathfrak{F}$. Thus, $G^{\mathfrak{F}} \in \mathfrak{N}_{p'} \cap \mathfrak{N}_p = (1)$. So, $G \in \mathfrak{F}$, as required.

(1) $\Rightarrow$ (4). Assume that $\mathfrak{F}$ is $\mathfrak{N}_\omega$-saturated. Let $h$ be the minimal composition satellite of $\mathfrak{H} = \mathrm{cform}(\mathfrak{F})$. Let $\mathfrak{M} = CF_\omega(f)$ where $f$ is an $\omega$-composition satellite satisfying the following conditions:

1) $f(p) = h(p)$ if $p \in \omega$;

2) $f(S) = \mathfrak{F}$ if $S \in \mathfrak{J} \setminus \{C_p \mid p \in \omega\}$.

Inclusion $\mathfrak{F} \subseteq \mathfrak{M}$ is evident. Assume that the converse inclusion is false, and let $G$ be a group of minimal order in $\mathfrak{M} \setminus \mathfrak{F}$. Then $L = G^{\mathfrak{F}}$ is a unique minimal normal subgroup in $G$. If $L$ is not an abelian $\omega$-group, it follows from $G \in \mathfrak{M}$ and $c_G(L) = 1$ that $G \simeq G/c_G(L) \in \mathfrak{F}$. Therefore $L$ is an $p$-group for some $p \in \omega$, and we have $G/C^p(G) \in f(p) = h(p)$. Thus $G \in \mathfrak{H}$. Since (1) $\Rightarrow$ (3), we get $G \in \mathfrak{N}_{p'}\mathfrak{F}$, and therefore $G^{\mathfrak{F}} \in \mathfrak{N}_{p'} \cap \mathfrak{N}_p = (1)$. So $\mathfrak{F} = \mathfrak{M}$. We notice that by Lemma 4.2 we have $f(p) = h(p) = \varnothing$ if $p \in \omega$ and $C_p \notin \mathrm{Com}(\mathfrak{F})$.

(4) $\Rightarrow$ (1). Let $G/L \in \mathfrak{F}$ and $L = \Phi(O_p(G))$, $p \in \omega$. By Lemma 2.8, $C^p(G)/L = C^p(G/L)$. Applying Proposition 4.1 to $G/L$, we have $G/O_p(G) \simeq (G/L)/O_p((G/L)) \in \mathfrak{F}$ and $G/C^p(G) \simeq (G/L)/C^p(G/L) = (G/L)/C^p(G/L) \in f(p)$. But then by Proposition 4.1 we get $G \in \mathfrak{F}$. $\square$

**Corollary 4.5.1.** *If a non-empty formation $\mathfrak{F}$ is $p$-solubly saturated and $C_p \in \mathrm{Com}(\mathfrak{F})$, then $\mathfrak{F}$ has an $p$-composition satellite $f$ such that $f(p') = \mathfrak{F}$ and $f(p) = \mathrm{Q}(G/C^p(G) \mid G \in \mathfrak{F})$.*

**Corollary 4.5.2.** *If a non-empty formation $\mathfrak{F}$ is solubly saturated, then $\mathfrak{F} = CF(f)$ where $f$ is a composition satellite satisfying the following conditions:*

*(i) $f(p) = \mathrm{Q}(G/C^p(G) \mid G \in \mathfrak{F})$ if $p \in \omega$ and $C_p \in \mathrm{Com}(\mathfrak{F})$;*

*(ii) $f(S) = \mathfrak{F}$ for every $S \in \mathrm{Com}^-(\mathfrak{F})$;*

*(iii) $f(S) = \varnothing$ for every $S \in \mathfrak{J} \setminus \mathrm{Com}(\mathfrak{F})$.*

**Theorem 4.6** (see [2], Theorem 3.1(b)). *Let $\mathfrak{F}$ be a non-empty $\omega$-saturated formation, and $h$ be the minimal composition satellite of* $\mathrm{cform}(\mathfrak{F})$. *Then $\mathfrak{F} = LF_\omega(f)$ where $f$ is an $\omega$-local satellite such that $f(p) = h(p)$ for every $p \in \omega$.*

**Proof.** We may suppose without loss of generality that $\omega \subseteq \pi(\mathfrak{F})$. By Lemma 4.2, $h(S) = \mathrm{Q}(H/C^S(H) \mid H \in \mathfrak{F})$ if $S \in \mathrm{Com}(\mathfrak{F})$, and $h(S) = \varnothing$ if $S \in \mathfrak{J} \setminus \mathrm{Com}(\mathfrak{F})$.

Let $p$ be a prime in $\omega$, and $S$ be a non-abelian $pd$-group in $\mathrm{Com}(\mathfrak{F})$. We will now prove that $h(S) \subseteq h(p)$. Consider $R = H/C^S(H)$, $H \in \mathfrak{F}$. By Lemma 2.1, $C^S(H)$ is the largest normal subgroup not having composition factors isomorphic to $S$. Clearly, $O_{p',p}(R) = 1$. Let $A_p(R)$ be the $p$-Frattini module, i. e., the kernel of the universal Frattini, $p$-elementary $R$-extension:

$$1 \to A_p(R) \xrightarrow{\mu} E \xrightarrow{\varepsilon} R \to 1.$$

Here $E/A_p(R) \simeq R$, and $A_p(R)$ is an elementary abelian $p$-group contained in $\Phi(E)$. Let $N_1, \dots, N_t$ be all minimal normal subgroups in $E$ contained in $A_p(R)$. Since $\mathfrak{F}$ is $p$-saturated, we have $E \in \mathfrak{F} \subseteq \mathrm{cform}(\mathfrak{F})$, and therefore $E/\cap_i C_E(N_i) \in h(p)$. Since $N_1, \dots, N_t$ are simple submodules of the $\mathbb{F}_pR$-module $A_p(R)$, it follows that $R/\mathrm{Ker}(R \textit{ on } (N_1 \dots N_t)) \in h(p)$. By theorem of Griess and P. Schmid, $\mathrm{Ker}(R \textit{ on } (N_1 \dots N_t)) = O_{p',p}(R)$ (see [17] or [4], p. 833). Since $O_{p',p}(R) = 1$, it follows that $R \in h(p)$. Thus, $h(S) = \mathrm{Q}(H/C^S(H) \mid H \in \mathfrak{F}) \subseteq h(p)$ if $S \in \mathrm{Com}(\mathfrak{F})$ and $p \in \omega \cap \pi(S)$.

Let $f$ be an $\omega$-local satellite such that $f(p) = h(p)$ if $p \in \omega$, and $f(\omega') = \mathfrak{F}$ if $\omega' \neq \varnothing$. We will prove now that $\mathfrak{F} = LF_\omega(f)$.

Let $G$ be a group of minimal order in $\mathfrak{F} \setminus LF_\omega(f)$. Then $L = G^{LF_\omega(f)}$ is a unique minimal normal subgroup in $G$, and $L$ is not $f$-central in $G$. If $L$ is an $\omega'$-group, then $c_G(L) = 1$ and $G \simeq G/c_G(L) \in f(\omega') = \mathfrak{F}$. If $L$ is a non-abelian $pd$-group for some $p \in \omega$ and $S \in \mathrm{Com}(L)$, then $C_G(L) = 1$ and we have $G \simeq G/C_G(L) \in h(S) \subseteq h(p) \subseteq \mathfrak{F}$. Assume that $L$ is a $p$-group, $p \in \omega$. Since $L$ is not $f$-central, $L \nsubseteq Z(G)$. By Lemma 2.1 we have $C^p(G) = 1$. So $G \in h(p) = \mathrm{Q}(H/C^p(H) \mid H \in \mathfrak{F})$, i. e., $L$ is $f$-central, a contradiction. Thus, $\mathfrak{F} \subseteq LF_\omega(f)$.

Let $G$ be a group of minimal order in $LF_\omega(f) \setminus \mathfrak{F}$. Then $L = G^{\mathfrak{F}}$ is a unique minimal normal subgroup in $G$. Clearly, $c_G(L) = 1$, and $C_G(L) = 1$ if $L$ is non-abelian. Hence, it follows from $G \in LF_\omega(f)$ that if $L$ is an $\omega'$-group, then $G \in f(\omega') = \mathfrak{F}$, and if $L$ is a non-abelian $pd$-group for some $p \in \omega$, then $G \in f(p) = h(p) \subseteq \mathfrak{F}$, and we get a contradiction. Assume that $L$ is a $p$-group, $p \in \omega$. Evidently, $L$ is not contained in $\Phi(G)$ (recall that $\mathfrak{F}$ is $p$-saturated). By Lemma 2.7, $L = C_G(L)$. Since $L$ is $f$-central, we obtain that $G = [L]T$ where $T \in f(p)$. Therefore, $T \simeq R/K$ where $R = H/C^p(H)$ for some $H \in \mathfrak{F}$. Now we can consider $L$ as an irreducible $\mathbb{F}_pR$-module by inflation (see [4], p. 105). By Lemma 2.5 we have $[L]R \in \mathfrak{F}$. Since $K$ acts identically on $L$, it follows from Lemma 2.6 that $[L](R/K) \simeq LT = G \in \mathfrak{F}$, and we again arrive at a contradiction. So $LF_\omega(f) = \mathfrak{F}$. $\square$

**Corollary 4.6.1.** *If a non-empty formation $\mathfrak{F}$ is $\omega$-saturated, then $\mathfrak{F}$ has an $\omega$-local satellite $f$ such that $f(p) = \mathrm{Q}(G/C^p(G) \mid G \in \mathfrak{F})$ if $p \in \omega \cap \pi(\mathfrak{F})$, $f(p) = \varnothing$ if $p \in \omega \setminus \pi(\mathfrak{F})$, and $f(\omega') = \mathfrak{F}$ if $\omega' \neq \varnothing$.*

**Corollary 4.6.2.** *If a non-empty formation $\mathfrak{F}$ is saturated, then $\mathfrak{F} = LF(f)$ where $f$ is a local satellite such that $f(p) = \mathrm{Q}(G/C^p(G) \mid G \in \mathfrak{F})$ for every $p \in \pi(\mathfrak{F})$, and $f(p) = \varnothing$ for every prime $p \notin \pi(\mathfrak{F})$.*

### 4. $\mathfrak{X}$-local formations

In 1985 Förster [18] introduced the concept '$\mathfrak{X}$-local formation' in order to obtain a common extension of Theorem 3.2 and 4.1.

**Definition 5.1.** Let $\mathfrak{X}$ be a class of simple groups such that $\mathrm{Char}(\mathfrak{X}) = \pi(\mathfrak{X})$. Consider a map

$$f : \pi(\mathfrak{X}) \cup \mathfrak{X}' \longrightarrow \{\text{formations}\}$$

which does not distinguish between any two non-identity isomorphic groups. Denote through $LF_{\mathfrak{X}}(f)$ the class of all groups $G$ satisfying the following conditions:

(i) if $H/K$ is a chief $\mathrm{E}\mathfrak{X}$-factor of a group $G$, then $G/C_G(H/K)$ belongs to $f(p)$ for any $p \in \pi(H/K)$;

(ii) if $G/L$ is a monolithic quotient of $G$ and $\mathrm{Soc}(G/L) \in \mathrm{E}(\mathfrak{X}')$, then $G/L \in f(S)$ where $S \in \mathrm{Com}(\mathrm{Soc}(G/L))$.

The class $LF_{\mathfrak{X}}(f)$ is a formation; it is called an $\mathfrak{X}$*-local formation*.

$\mathfrak{X}$-local formations were investigated in [20, 19, 21, 22, 23]. In [24] it was proved with help of some lemmas in [22] that every $\mathfrak{X}$-local formation has a $\mathfrak{X}^+$-composition satellite. Now we give a direct proof of that fact.

**Theorem 5.1.** *Let $\mathfrak{F}$ be a non-empty formation, $\mathfrak{X}$ a class of simple groups such that* $\mathrm{Char}(\mathfrak{X}) = \pi(\mathfrak{X})$. *Let $\mathfrak{L}$ be a class of simple groups such that $\mathfrak{L}^+ = \mathfrak{X}^+$.*

(1) *If $\mathfrak{F}$ is an $\mathfrak{X}$-local formation, then $\mathfrak{F}$ has an $\mathfrak{L}$-composition satellite.*

(2) *If $\mathfrak{F}$ has an $\mathfrak{L}$-composition satellite, then $\mathfrak{F}$ is an $\mathfrak{X}^+$-local formation.*

**Proof.** Set $\omega = \mathrm{Char}(\mathfrak{X})$. Evidently, $\mathfrak{L}^- \cup \mathfrak{L}' = \mathfrak{X}^- \cup \mathfrak{X}' = (\mathfrak{L}^+)' = (\mathfrak{X}^+)'$.

(1) Let $\mathfrak{F}$ be a $\mathfrak{X}$-local formation, $\mathfrak{F} = LF_{\mathfrak{X}}(f)$. Consider an $\mathfrak{L}$-composition satellite $h$ such that $h(p) = f(p) \cap \mathfrak{F}$ if $p \in \omega$, and $h(S) = \mathfrak{F}$ if $S \in \mathfrak{L}^- \cup \mathfrak{L}'$. We will prove that $\mathfrak{F} = CF_{\mathfrak{L}}(h)$.

Suppose that $\mathfrak{F} \not\subseteq CF_{\mathfrak{L}}(h)$. Let $G$ be a group of minimal order in $\mathfrak{F} \setminus CF_{\mathfrak{L}}(h)$. Then $G$ is monolithic, and $G/M \in CF_{\mathfrak{L}}(h)$ where $M$ is the socle of $G$. Clearly, $M$ is the $CF_{\mathfrak{L}}(h)$-residual of $G$, and every chief factor between $G$ and $L$ is $h$-central. Assume that $M$ is an $\mathrm{E}(\mathfrak{L}^- \cup \mathfrak{L}')$-group. Since $G \in \mathfrak{F}$, we have that $G \in h(S)$ where $S \in \mathrm{Com}(M)$. Since $c_G(L) = 1$, we have that $M$ is $h$-central in $G$, and so $G \in CF_{\mathfrak{L}}(h)$. Assume now that $M$ is a $p$-group, $p \in \omega$. Since $G \in \mathfrak{F}$, we have $G/C_G(M) \in f(p) \cap \mathfrak{F} = h(p)$, i. e., $M$ is $h$-central. We see that $G \in CF_{\mathfrak{L}}(h)$, a contradiction. Thus, $\mathfrak{F} \subseteq CF_{\mathfrak{L}}(h)$.

Suppose now that $CF_{\mathfrak{L}}(h) \not\subseteq \mathfrak{F}$. Choose a group $G$ of minimal order in $CF_{\mathfrak{L}}(h) \setminus \mathfrak{F}$. Then $G$ is monolithic, and $G/M \in \mathfrak{F}$ where $M = G^{\mathfrak{F}}$ is the socle of $G$. Assume that $M$ is an $\mathrm{E}(\mathfrak{L}^- \cup \mathfrak{L}')$-group. Then from $c_G(L) = 1$ and $h$-centrality of $L$ it follows that $G/c_G(M) \simeq G \in \mathfrak{F}$. Assume that $M$ is a $p$-group, $p \in \omega$. Then

$$G/C_G(M) \in h(p) = \mathfrak{F} \cap f(p) \subseteq f(p).$$

We see that all the chief factors and all the quotients of $G$ satisfies conditions (1) and (2) of Definition 5.1. So, $G \in \mathfrak{F}$, a contradiction. Thus, $\mathfrak{F} = CF_{\mathfrak{L}}(h)$.

(2) Let $\mathfrak{F}$ be a formation having an $\mathfrak{L}$-composition satellite. By Lemma 4.3, $\mathfrak{F} = CF_{\mathfrak{L}}(f)$ where $f$ is an $\mathfrak{L}$-composition satellite such that $f(S) = \mathfrak{F}$

for every $S \in \mathfrak{L}^- \cup \mathfrak{L}'$. Consider an $\mathfrak{X}^+$-local formation $\mathfrak{H} = LF_{\mathfrak{X}^+}(h)$ where $h(p) = f(p)$ for any $p \in \omega$, and $h(S) = \mathfrak{F}$ for every $S \in (\mathfrak{X}^+)'$. It easy to check that $\mathfrak{F} = \mathfrak{H}$. $\square$

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